BALL BEARING STIFFNESS.
A NEW APPROACH OFFERING ANALYTICAL EXPRESSIONS

Pascal GUAY[(1)], Ahmed FRIKHA[(2)]

[(1)] Airbus Defence & Space, 31 Rue des cosmonautes 31402 Toulouse Cedex 4, France, pascal.guay@airbus.com
[(2)] ADR, 12 Chemin des prés 77810 Thomery, France, afrikha@adr-alcen.com

ABSTRACT

Space mechanisms use preloaded ball bearings in order to withstand the severe vibrations during launch.
The launch strength requires the calculation of the bearing stiffness, but this calculation is complex.
Nowadays, there is no analytical expression that gives the stiffness of a bearing. Stiffness is computed using an iterative algorithm such as Newton-Raphson, to solve the nonlinear system of equations.
This paper aims at offering a simplified analytical approach, based on the assumption that the contact angle is constant. This approach gives analytical formulas of the stiffness of preloaded ball bearing.

Notations

a	Semimajor axis of contact ellipse
b	Semiminor axis of contact ellipse
B	$= f_e + f_i - 1$ Total curvature of the bearing
D	Ball diameter
d_m	Bearing pitch diameter
E	Modulus of elasticity
e	Axial deflection due to preload
f	$f_e = r_e / D$ Dimensionless parameter
F(κ)	Elliptic integral of the first kind
K_n	Ball stiffness
k_a	Axial stiffness of paired bearing
k_r	Radial stiffness of paired bearing
p_H	Hertzian pressure
P	Bearing preload
Q	Ball normal load
r_i r_e	Raceway groove curvature radius
R_x R_y	Equivalent curvature radius
S(κ)	Elliptic integral of the second kind
Z	Ball complement
α	Contact angle
δa	Axial deflection
δn	Normal approach along the line of contact
δr	Radial deflection
ε	$= 0,5\,[1+ (\delta a/\delta r) \tan\alpha]$
γ	$= D \cos\alpha / d_m$ Dimensionless parameter
Γ	Curvature difference
κ	$= a / b$ Elongation of elliptic contact area
ν	Poisson's ratio
ρ	$= R_y / R_x$

1 INTRODUCTION

Preloaded angular contact ball bearings are used in several applications, submitted to severe vibrations: spindles of machine tools, gyroscopes, and space or military mechanisms. The preload suppresses the backlash, which highly improves the strength to launch vibrations, but also offers pointing accuracy.
Mastering bearing stiffness allows to define the optimum preload level. A too low preload generates a high gapping under launch vibrations, which generates shocks that may damage balls and tracks. A too high preload generates a high friction torque and degrades the life duration.

Bearing stiffness calculation is usually done using an iterative algorithm such as Newton-Raphson, because the contact angle depends on the loading. There is no analytical solution giving the bearing stiffness.
The purpose of this paper is to provide with an analytical expression of ball bearing stiffness, for a preloaded paired bearing.

2 BALL STIFFNESS

The calculation of the ball stiffness is complex. It is based on Hertz theory [1]. Jones proposed in 1946 a simplified calculation [2] [7]. But it leads to an under-estimated ball stiffness by 5 to 10%.

2.1 Hertz theory

Under a normal load, the contact area between the ball and the ring is elliptic. The pressure manifold is a paraboloid. The maximum contact pressure is located at the centre of the elliptic area. It is called the Hertzian pressure and is given by following expression

$$p_H = \frac{3Q}{2\pi ab} \qquad (01)$$

with
a semi-major axis, b semi-minor axis
Q normal load

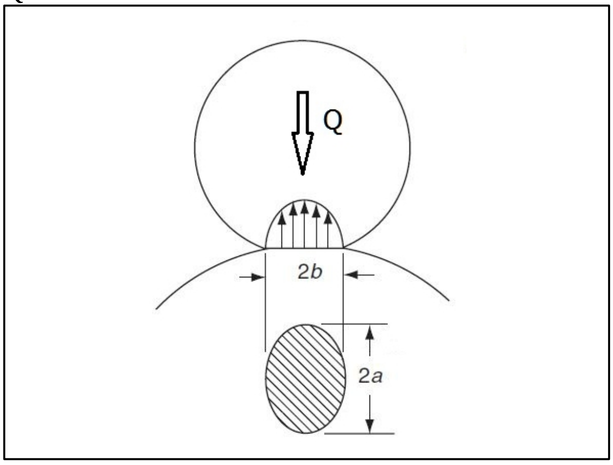


Figure 1 - Hertzian pressure and deformation

Curvature preliminary calculations

In a first step, a few curvature parameters of the contact must be calculated.
Let note:
x Lateral direction
y Rolling direction
D Ball diameter
r Raceway groove curvature radius
f_i f_e Raceway conformities $f_e = r_e / D$
γ $= D \cos\alpha / d_m$

Curvature radii must be calculated at first.

Table 1: Definition of curvature radii

	Inner raceway	Outer raceway
R_x	$R_{xi} = (1-\gamma)\ D/2$	$R_{xe} = (1+\gamma)\ D/2$
R_y	$R_{yi} = f_i\ D / (2f_i - 1)$	$R_{ye} = f_e\ D / (2f_e - 1)$

One can now calculate the equivalent radius R and the curvature difference Γ

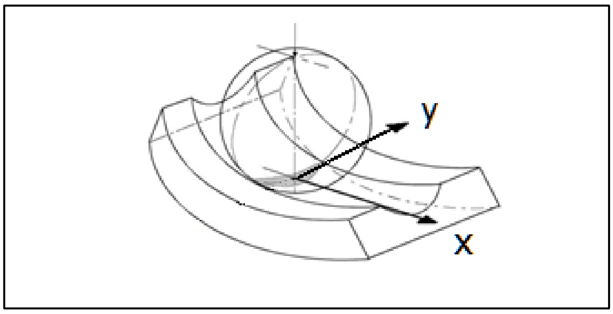
$$\frac{1}{R} = \frac{1}{R_x} + \frac{1}{R_y} \qquad (02)$$

$$\Gamma = R\ (\frac{1}{R_y} - \frac{1}{R_x}) \qquad (03)$$

The quantity 1/R may be used in other papers, and it is named the "curvature sum".

The elongation κ of the ellipse is the key parameter for the problem solving. It is defined as

$$\kappa = a / b \qquad (04)$$



Ball on raceway contact: κ > 1

Figure 2 – The elliptic contact area

κ depends on the curvatures of the two contacting bodies. It is obtained by solving the equation

$$1 - \frac{2}{\kappa^2 - 1}\left[\frac{F(\kappa)}{S(\kappa)} - 1\right] - \Gamma = 0 \qquad (05)$$

where F(κ) and S(κ) are respectively the elliptic integral of first kind and second kind

$$F(\kappa) = \int_0^{\frac{\pi}{2}} \frac{1}{\sqrt{1-\left(1-\frac{1}{\kappa^2}\right)\sin^2\psi}}\, d\psi \qquad (06)$$

$$S(\kappa) = \int_0^{\frac{\pi}{2}} \sqrt{1-\left(1-\frac{1}{\kappa^2}\right)\sin^2\psi}\ d\psi \qquad (07)$$

There is no general analytical solution to this problem. The elastic contact problem between two balls, as studied by Hertz in 1880, is still nowadays the only existing analytical solution [1].

The semimajor axis of the ellipse is given by

$$a = \left[\frac{6\kappa^2\ S(\kappa)\ Q\ R}{\pi\ E}\right]^{1/3} \qquad (08)$$

with
Q Normal load
E Equivalent modulus of elasticity given by

$$E = \frac{2}{\frac{1-\nu_1^2}{E_1} + \frac{1-\nu_2^2}{E_2}} \qquad (09)$$

with
E_j modulus of elasticity of body j
ν_j Poisson's ratio of body j.

Once the elongation is known, the semiminor axis is obtained from (04) b = a / κ

The formula (12) given in ref [5] was erronous. The corrected one is F(κ) = π /2 + q ln(ρ)

2.2 Hamrock and Anderson solution

In 1973, Hamrock and Anderson used a numerical procedure with the method of least squares and obtained simplified approximations allowing to solve Hertz theory in any case [3] [5]. They are the following:

Ellipse elongation:

$$\kappa \approx \rho^{2/\pi} \qquad (10)$$

Elliptic integral of the 2nd kind:

$$S(\kappa) \approx 1 + \frac{q}{\rho} \qquad (11)$$

Elliptic integral of the 1st kind

$$F(\kappa) \approx \frac{\pi}{2} + q(1 + ln\rho) \qquad (12)$$

with

ρ = Ry / Rx

$q = \pi/2 - 1$

The parameter ρ is always superior to 1 in the case of ball bearings.

The error on the function κ(ρ) is 3,8% on the range $1 < \rho < 100$. It is 2,1% on F(κ) and 0,9% on S(κ).
Of course, these functions can be refined further.
The authors of this paper found the following curve fit with 1% error

$$\kappa \approx 1{,}18\ \rho^{0{,}598} - 0{,}19 \qquad (13)$$

2.3 Contact deformation

The contact deformation is given by Hertz theory [1].
On the ball /outer raceway contact, the relative approach of the contacting bodies is [5] [8]:

$$\delta_e = \frac{F_e(\kappa_e)}{\left[2S_e(\kappa_e)R_e\right]^{1/3}} \left(\frac{3Q}{\pi\kappa_e E}\right)^{2/3} \qquad (14)$$

with
index e for “external raceway”
index i for “internal raceway”.

The same equation is valid on ball /internal ring contact, just replacing index e by index i. The deformation is maximum at the center of the contact area.

The stiffness along the normal to the contact is not linear. It increases with the load because the elliptic contact area increases. From the previous equation, one obtains:

$$Q = K_i\, \delta_i^{3/2} = K_e\, \delta_e^{3/2} \qquad (15)$$

with K_i, K_e Stiffness of ball/raceway contact

$$K_e = \frac{\pi}{3} \kappa_e E \sqrt{\frac{2S_e(\kappa_e)R_e}{\left[F_e(\kappa_e)\right]^3}} \qquad (16)$$

The same equation is valid for K_i, replacing index e by index i.

2.4 Coupling the two ball/raceway contacts

Let now express the ball stiffness:

$$Q = K_n\, \delta n^{3/2} \qquad (17)$$

with
δn Relative approach of the two rings along the normal to the contact.
K_n Ball stiffness along the normal

The two deformations are to be added to express the relative approach of the opposite raceways:

$$\delta n = \delta i + \delta e \qquad (18)$$

Using equation (15), one obtains:

$$\delta_n = Q^{2/3} \cdot \left(\frac{1}{K_i^{2/3}} + \frac{1}{K_e^{2/3}} \right) \qquad (19)$$

Finally, the ball stiffness expression is [07]:

$$K_n = \left(\frac{1}{K_i^{2/3}} + \frac{1}{K_e^{2/3}}\right)^{-3/2} \qquad (20)$$

The ball stiffness K_n is not linear but it does not depend on the load.

2.5 Constant contact angle hypothesis

In space applications, ball bearings usually have a high contact angle superior to 25°. In this case, the contact angle increase under preload is small (typically 0,3 to 0,5°). So the contact angle increase can be neglected for stiffness calculations.

This simplifies considerably the calculations, and allows obtaining analytical expressions of the bearing stiffness around its preloaded state.

3 AXIAL STIFFNESS

3.1 Single bearing

Axial deflection

Let now consider a single ball bearing submitted to a centered axial load F_a. The axial load F_a is equally distributed on all the balls, and it generates on each ball a normal load Q

$$Q = \frac{F_a}{Z \sin\alpha} \qquad (21)$$

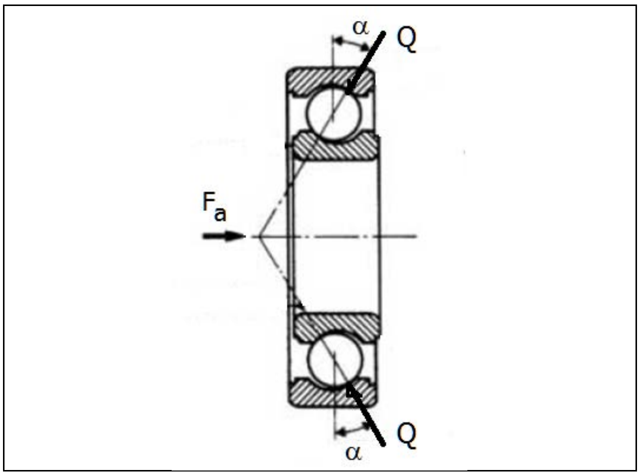


Figure 3 - Axial loading of an angular contact bearing

Substituting (17) into (21) yields:

$$F_a = ZK_n \sin\alpha\ \delta n^{3/2} \qquad (22)$$

As the contact angle is considered constant, the relation between normal deflection δn and axial deflection δa takes a simple form [8]:

$$\delta n = \delta a \sin\alpha \qquad (23)$$

One obtains the equation of the axial deflection curve:

$$F_a = ZK_n \sin\alpha^{5/2}\ \delta a^{3/2} \qquad (24)$$

Axial stiffness

The axial stiffness of the single ball bearing around its preloaded state has for expression

$$k_a = \left[\frac{dF_a}{d\delta_a}\right]_{Fa=P} = \frac{3}{2} ZK_n \sin\alpha^{5/2}\ \delta a_p^{1/2} \qquad (25)$$

with

$\delta a_p = e/2$ axial deflection under the pure preload

P Preload

From equation (24), the axial deflection can be expressed as a function of the preload

$$\delta a_p = [\ P/(Z K_n \sin\alpha^{5/2})\ ]^{2/3} \qquad (26)$$

Substituting this expression into (17), one obtains the axial stiffness of a single ball bearing

$$k_{a\ single} = 1{,}5\ (Z K_n)^{2/3} \sin\alpha^{5/3}\ P^{1/3} \qquad (27)$$

The axial stiffness is sensitive to the contact angle, but not so much to the preload.

3.2 Paired bearing

Let consider the usual case of two angular contact ball bearings back-to-back mounted with a rigid preload P.
An external axial load A is applied on the paired bearing.

Axial equilibrium of the bearing

When the external load A is zero, each single row sees an axial load equal to the preload:

$$F_{a1} = F_{a2} = P \qquad (28)$$

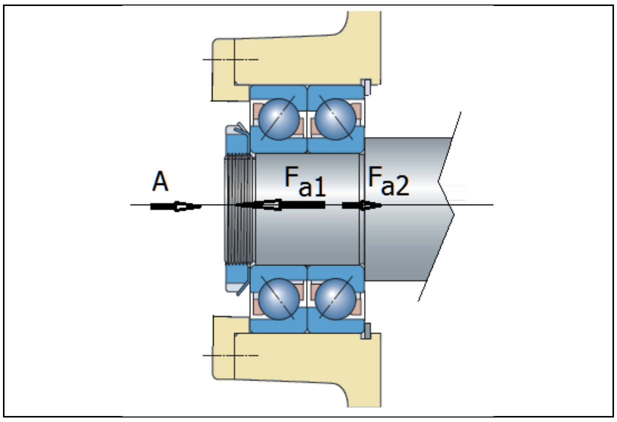


Figure 4 – Preloaded bearing submitted to an axial load

When a positive axial load A is applied, the mechanical equilibrium of the bearing is written

$$A = F_{a1} - F_{a2} \qquad (29)$$

with

F_{a1} axial load on single bearing 1

F_{a2} axial load on single bearing 2

Deflection curve

The deflection curve of a paired bearing shows the evolution of the load distribution F_{a1} and F_{a2} as a function of the external load A (**Figure 5**).

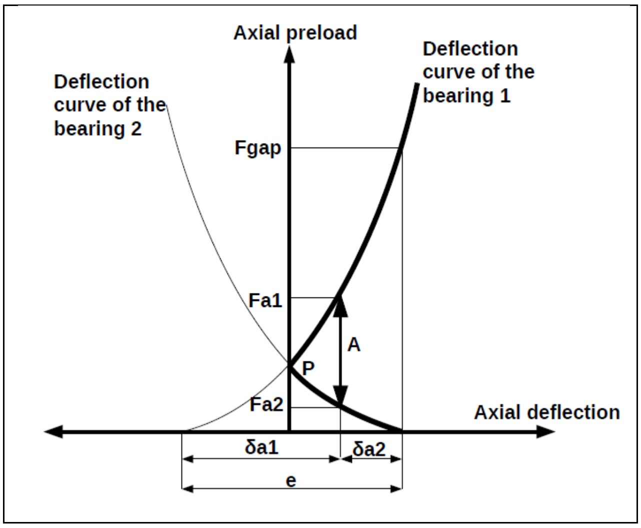


Figure 5 – Deflection curve of a paired bearing

The preload P corresponds to the intersection point of the two single curves in opposition. It generates an initial deflection δa_1 on bearing 1 and δa_2 on bearing 2, which satisfies the relation

$$\delta a_1 + \delta a_2 = e \quad (30)$$

with e Relative approach of the rings under preload

In the usual case of two identical single bearings assembled with a rigid preload, the following relation is obtained from (24):

$$P = Z\, K_n \sin\alpha^{\,5/2}\, (e/2)^{3/2} \quad (31)$$

Gapping threshold

When the axial deflection is such as δa = e, the gapping threshold is reached. The single bearing 1 withstands all the loading and the single bearing 2 gets offloaded. The gapping load is

$$F_{gap} = Z\, K_n \sin\alpha^{\,5/2}\, e^{3/2} = 2^{3/2}\, P \approx 2{,}83\, P \quad (32)$$

One finds the theoretic justification of a well-known formula. Beyond this threshold, only the single bearing 1 is loaded.

Stiffness of paired bearing

The axial stiffness of the paired bearing is the sum of the axial stiffness of each single bearing.
When the single bearings are identical, it is worth twice the stiffness of the single bearing:

$$k_a = 2\, k_{a\ single} \quad (33)$$

$$k_a = 3\, (Z\, K_n)^{2/3} \sin\alpha^{5/3}\, P^{\,1/3} \quad (34)$$

This stiffness can be expressed as a function of the preload parameters P and e.
Using equations (25) and (31) it comes

$$\frac{k_a}{P} = \frac{3ZK_n \sin\alpha^{5/2} \delta a_p^{1/2}}{ZK_n \sin\alpha^{5/2} \delta a_p^{3/2}} = \frac{3}{\delta a_p} = \frac{6}{e} \quad (35)$$

hence

$$k_a = 6P/e \quad (36)$$

4 RADIAL STIFFNESS

One considers as formerly a paired bearing comprising two identical single ball bearings mounted in opposition with a rigid preload P. A pure and centred radial load is applied on the paired bearing. In these conditions, the radial load generates a motion of pure radial translation in the paired bearing [6] [8].

4.1 Radial deflection

Starting from the initial preloaded state, an external increasing radial load is applied on the paired bearing. The paired bearing comprises two mobile rings and two fixed rings. Under the load, the two mobile rings achieve the radial displacement δr. But the fixed axial deflection δa = e /2 due to preload remains installed.

One defines the parameter ε that is linked to the number of loaded balls (see Figure 6):

$$\varepsilon = \frac{1}{2}\left(1 + \frac{\delta a}{\delta r} \tan\alpha\right) \quad (37)$$

with
δa = e /2 Axial deflection of mobile rings
δr Radial deflection of mobile rings
α Contact angle

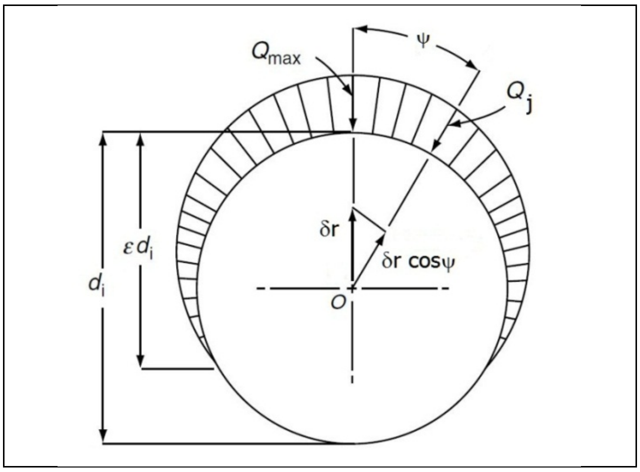


Figure 6 – Loaded zone

The loading zone is symmetrical and its range is $-\psi_o$ to $+\psi_o$ with

$$\psi_o = \text{acos}(1 - 2\varepsilon) \text{ if } \varepsilon < 1$$

$$\psi_o = \pi \quad \text{otherwise} \qquad (38)$$

Beyond this zone, the ball load Q is zero.

When the radial load applied is small, δr approaches zero and ε becomes infinite.
When the radial load is important and approaches the static capability of the bearing, δa /δr approaches zero and ε approaches the value 0,5.

Here the rings are considered as rigid bodies that only have an elastic deformation below the ball contact. When the bearing is loaded, the normal load on the most loaded ball can be written [8]

$$Q_{max} = K_n (2\varepsilon\ \delta r \cos\alpha)^{3/2} \qquad (39)$$

with K_n Ball stiffness

The centered radial load applied on the paired bearing is noted 2 F_r. So, each ball row sees a load F_r. The relation between the radial load F_r acting on a single row and the ball load Q_{max} is [8]

$$F_r = Z\, Q_{max} \cos\alpha\, J_r(\varepsilon) \qquad (40)$$

with
Z Number of balls per row (ball complement)
J_r Radial integral

$$J_r(\varepsilon) = \frac{1}{2\pi}\int_{-\psi_o}^{+\psi_o}[1-\frac{1}{2\varepsilon}(1-\cos\psi)]^{3/2}\cos\psi.d\psi \qquad (41)$$

Combining the two last equations, one obtains the equations of the radial deflection curve:

$$F_r = Z\, K_n\, J_r(\varepsilon) \cos\alpha^{5/2} (2\varepsilon\ \delta r)^{3/2} \qquad (42)$$

But here the number of loaded balls and the parameter ε depend on the load.

4.2 Radial stiffness

On the paired bearing, the radial stiffness around the preloaded state is given by following equation:

$$k_r = \frac{d}{d\delta r}[2F_r]\ \delta r \to 0$$

$$= 2ZK_n \cos\alpha^{5/2} \frac{d}{d\delta r}\left[J_r(\varepsilon)(2\varepsilon\delta r)^{3/2}\right] \qquad (43)$$

Around the preloaded state, one has δa =e/2 and δr approaches 0, hence ε becomes infinite.
Equation (37) takes the form:

$$\varepsilon = \frac{1}{2}\left(1 + \frac{e \cdot \tan\alpha}{2\delta r}\right) \qquad (44)$$

When dr approaches 0, it comes:

$$2\varepsilon\ \delta r = (\delta r + e \tan\alpha /2) \approx e \tan\alpha /2 \qquad (45)$$

Consequently, when δr approaches 0, ε becomes infinite.

We must now calculate the limit of J_r when ε becomes infinite, knowing that:
a) when ε approaches infinite, all the balls are loaded, yielding $\psi_o = \pi$
b) the term between brackets $[1-\frac{1}{2\varepsilon}(1-\cos\psi)]^{3/2}$ can be written $[1-\frac{3}{4\varepsilon}(1-\cos\psi)]$
c) the function to be integrated is even.

When ε approaches infinite, it comes

$$\begin{aligned} J_r(\varepsilon) &= \frac{1}{\pi}\int_0^{\pi}[1-\frac{3}{4\varepsilon}(1-\cos\psi)]\cos\psi\, d\psi \\ &= \frac{1}{\pi}(1-\frac{3}{4\varepsilon})\int_0^{\pi}\cos\psi\, d\psi \\ &\quad + \frac{3}{4\pi\varepsilon}\int_0^{\pi}\cos^2\psi\, d\psi \\ &= 0 + \frac{3}{4\pi\varepsilon}\frac{\pi}{2} = \frac{3}{8\varepsilon} \end{aligned} \qquad (46)$$

hence

$$J_r(\varepsilon)(2\varepsilon\delta r)^{3/2} \approx \frac{3}{8\varepsilon}(\frac{e\tan\alpha}{2})^{3/2} \qquad (47)$$

Consequently

$$\frac{d}{d\delta r}\left[J_r(\varepsilon)(2\varepsilon\delta r)^{3/2}\right] \approx \frac{3}{8}\left(\frac{e\tan\alpha}{2}\right)^{3/2}\frac{d}{d\delta r}\left(\frac{1}{\varepsilon}\right) \quad (48)$$

with, from (45):

$$\frac{d}{d\delta r}\left(\frac{1}{\varepsilon}\right) = \frac{d}{d\delta r}\left(\frac{4\delta r}{e\tan\alpha}\right) = \frac{4}{e\tan\alpha} \quad (49)$$

Finally, the radial stiffness of the paired bearing is

$$k_r = 3/(2\sqrt{2})\ \ Z\, K_n \cos\alpha^{5/2}\ \ (e \tan\alpha)^{1/2} \quad (50)$$

Let compare this expression to this of axial stiffness.
For the paired bearing, eq. (25) and (33) give

$$k_a = 3\ Z\ K_n \sin\alpha^{5/2}\ (e/2)^{1/2} \quad (51)$$

Finally

$$k_r = \frac{k_a}{2\tan^2\alpha} \quad (52)$$

4.3 Contact angle which offers the same axial and radial stiffness

The equation (52) allows calculating the contact angle which offers the same axial and radial stiffness. Axial and radial stiffness are equal when $2\tan^2\alpha = 1$, that is to say

$$\alpha = 35{,}26°$$

5 EXAMPLE

SPOT 5 MCV paired bearing. The paired ball bearing is mounted with a rigid preload.

Pitch diameter	d_m = 108 mm
Ball diameter	D = 6,35 mm
Ball complement	Z = 40
Initial contact angle	α_o = 36°
Total curvature	B = 0,075 (7% and 8%)
Preload	P = 560 N
Ball stiffness	K_n = 213 033 N/mm3/2

The exact calculations were computed on ADR software. They are compared to the approached calculation in Tab. 2:

Table 2 - Comparison of calc. results

	Contact angle	**Axial stiffness k_a**	**Radial stiffness k_r**
Approx calc.	36°	426 N/µm	403 N/µm
Exact calc.	36,38°	432 N/µm	395 N/µm
Error	0,38°	1,0 %	2,0 %

On the SPOT5 MCV mechanism, the approximate results are excellent with a small error of 2%.

6 CONCLUSION

Nowadays, the ball bearings are computed with specific softwares, whose use is not so easy for the engineers, because the definition of a bearing remains much more complex than expected. Static capability calculation and life duration can be made by hand knowing the static and dynamic basic ratings, but there is no available formula giving the bearing stiffness.
Assuming that the contact angle is constant to derive the stiffness, this paper answers to this lack, obtaining stiffness analytical expressions of preloaded ball bearings. These new expressions can help in designing space mechanisms.

REFERENCES

1. HERTZ H. (1882). Le mémoire de Hertz sur les contacts ponctuels, ENSAM Paris 1985, *Publication scientifique et technique* n° 30 - (Original release «Uber die Berührung fester elastischer Körper und über die Härte, Verhandlungen des Vereins zur Beförderung des Gewerbefleisses, 1882, p 449-463)

2. JONES A. (1946). Analysis of Stresses and Deflections, New Departure Engineering Data, Bristol, CT.

3. HAMROCK B.J. and ANDERSON W.J (1973). Analysis of an arched outer-race ball bearing considering centrifugal forces, *ASME Journal of Lubrication technology*, vol. 95, no 3, pp265-276.

4. HAMROCK B.J. and DOWSON D. (1981). Ball Bearing Mechanics, *NASA Technical Memorandum* 81691, 105p.

5. HAMROCK B.J. and ANDERSON W.J (1983). Rolling-element bearings, *NASA Reference publication* Volume 1105, 57p.

6. AUBLIN M., BONCOMPAIN R., BOULATON M., CARON D., JEAY E., LACAGE B. et REA J. (1988) Systèmes mécaniques. Théorie et dimensionnement. *Dunod, Paris,* ISBN 2 10 004162 2, 665 p.

7. HARRIS T.A. (2001). Rolling bearing analysis, 4th edition, *John Wiley & Sons*. *New York*, 1105 p.

8. GUAY P. (2013). BM 5371 Roulements. Calculs, *Techniques de l'ingénieur*, 39p.